MATH 141: Midterm 2

Name: key

Directions:

* Show your thought process (commonly said as ”show your work”) when solving each problem for full credit.
* **Remember to simplify each expression.**
* If you do not know how to solve a problem, try your best and/or explain in English what you would do.
* Good luck!

Problem	Score	Points
1		10
2		10
3		10
4		10
5		10
6		10
		60

1. Suppose $f(x) = \sqrt{x}$.

(a) What does the expression $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ represent?

The derivative of $f(x)$, which represents the slope of the tangent line at the same x-coordinates.

(b) Find

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for the given function $f(x)$. You must use this limit definition to receive credit.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$A^2 - B^2$

$$= \lim_{h \to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)}$$

frac law 5

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

continuity

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \boxed{\frac{1}{2\sqrt{x}}}$$

(c) Find the equation of the tangent line of $f(x)$ at the point $(1, 1)$.

Tangent line of $f(x)$ at $(a, f(a))$ is

$$y - f(a) = f'(a)(x - a)$$

Given $(1, 1)$, we have $y - 1 = f'(1)(x - 1)$

a $f(a)$

$$y - 1 = \frac{1}{2\sqrt{1}}(x - 1)$$

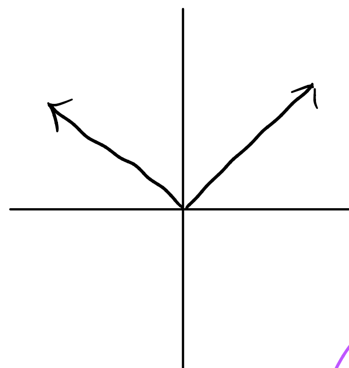
$$y = 1 + \frac{1}{2}x - \frac{1}{2}$$

$$\boxed{y = \frac{1}{2}x + \frac{1}{2}}$$

2. Short answer questions:

(a) If a function $f(x)$ is continuous at $x = a$, must it be differentiable at $x = a$ as well? If not, draw a graph of a function that is continuous but not differentiable at $x = a$.

No. The function $f(x) = |x|$ has the following graph.



$f(x)$ is continuous on $\mathbb{R}$ but differentiable on $(-\infty, 0) \cup (0, \infty)$.

(b) True or false:

$$f(x) = \sin(x) + \frac{x}{x+1}$$

is continuous on $\mathbb{R}$.

False. Finding continuity is just finding domain in Calculus 1.

$\frac{x}{x+1}$ has domain $(-\infty, -1) \cup (-1, \infty)$ because you cannot divide by 0.

$\therefore$ $f(x)$ is continuous on $(-\infty, -1) \cup (-1, \infty)$.

(c) Given $f(x) = x$, find an equation of the normal line at $(3, 3)$.

a $f(a)$

$f'(x) = 1$

The normal line at $(a, f(a))$ is

$$y - f(a) = -\frac{1}{f'(a)}(x - a) \quad \text{so}$$

$$y - 3 = -\frac{1}{1}(x - 3)$$

$$y = 3 - x + 3$$

$$\boxed{y = -x + 6}$$

3. Answer the following:

(a) Given a function $f(x)$, if

$$\lim_{x\to a} f(x) = \frac{0}{0}$$

what global factor do you need to manifest in the numerator and denominator and why?

the factor $x-a$ needs to be created in order to use fraction law 5 to cancel, removing the 0's from the numerator and denominator.

(b) Find

$$\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t}$$

create global factor of t and cancel.

Try limit laws:

$$\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} = \frac{\sqrt{1+0}-\sqrt{1-0}}{0} = \frac{1-1}{0} = \boxed{\frac{0}{0}}$$

Precalc to create factor of t:

$$\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} \cdot \frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}} \overset{A^2-B^2}{=} \lim_{t\to 0} \frac{\overbrace{1+t}^{A^2} - \overbrace{(1-t)}^{B^2}}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}$$

$$= \lim_{t\to 0} \frac{1+t-1+t}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}$$

$$= \lim_{t\to 0} \frac{2\cancel{t}}{\cancel{t}\left(\sqrt{1+t}+\sqrt{1-t}\right)}$$

$$\overset{\text{frac law 5}}{=} \lim_{t\to 0} \frac{2}{\sqrt{1+t}+\sqrt{1-t}}$$

$$\overset{\text{continuity}}{=} \frac{2}{\sqrt{1+0}+\sqrt{1-0}} = \frac{2}{1+1}$$

$$= \boxed{1}$$

(c) Find

$$\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}$$

create global factor of $x-3$ and cancel.

Try limit laws:

$$\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}=\frac{\frac{1}{3}-\frac{1}{3}}{3-3}=\boxed{\frac{0}{0}}$$

Precalc to create $x-3$ in numerator:

$$\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}\cdot\frac{3x}{3x}\overset{\text{frac law}}{=}\lim_{x\to 3}\frac{\left(\frac{1}{x}-\frac{1}{3}\right)3x}{(x-3)\cdot 3x}$$

$$\overset{\text{dist}}{=}\lim_{x\to 3}\frac{\frac{1}{\cancel{x}}\cdot 3\cancel{x}-\frac{1}{\cancel{3}}\cdot\cancel{3}x}{3x(x-3)}$$

frac law 5 to get rid of compound fraction

$$=\lim_{x\to 3}\frac{3-x}{3x(x-3)}$$

$$=\lim_{x\to 3}\frac{-x+3}{3x(x-3)}$$

negative law 2

$$=\lim_{x\to 3}\frac{-x-(-3)}{3x(x-3)}$$

GCF

$$=\lim_{x\to 3}\frac{-\cancel{(x-3)}}{3x\cancel{(x-3)}}$$

frac law 5

$$=\lim_{x\to 3}\frac{-1}{3x}$$

$$=\boxed{-\frac{1}{9}}$$

4. Find the following derivatives. You are allowed to use the Differentiation Rules.

(a) $f(x) = \pi^2$

constant

$$f'(x) = \frac{d}{dx}\, \pi^2 = 0$$

(b) $f(x) = \overbrace{x^2}^{\text{left}} \cdot \overbrace{\sin x}^{\text{right}}$ Product Rule

$$f'(x) = x^2 \frac{d}{dx}\left[\sin(x)\right] + \sin(x) \cdot \frac{d}{dx}\left[x^2\right]$$

$$= \boxed{x^2 \cos(x) + 2x \sin(x)}$$

(c) $f(x) = \dfrac{\overbrace{\sin(x^2)}^{\text{top}}}{\underbrace{2 - \cos x}_{\text{bottom}}}$ Quotient Rule

$$f'(x) = \frac{(2-\cos(x)) \cdot \overbrace{\frac{d}{dx}\sin(x^2)}^{\text{Chain rule}} - \sin(x^2)\frac{d}{dx}\left[2 - \cos(x)\right]}{(2-\cos(x))^2}$$

$$= \frac{(2-\cos(x)) \cdot \cos(x^2) \cdot \frac{d}{dx} x^2 - \sin(x^2) \cdot (0 - (-\sin(x)))}{(2-\cos(x))^2}$$

$$= \frac{(2-\cos(x)) \cdot \cos(x^2) \cdot 2x - \sin(x^2)\sin(x)}{(2-\cos(x))^2}$$

$$= \boxed{\frac{2\cos(x^2) - \cos(x)\cos(x^2) - \sin(x^2)\sin(x)}{(2-\cos(x))^2}}$$

Three function composition

Chain Rule

(d) $g(x) = \sqrt{\tan x^3} = \left(\tan(x^3)\right)^{\frac{1}{2}}$

$$g'(x) = \frac{1}{2}\left(\tan(x^3)\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\tan(x^3)$$

$$= \frac{1}{2}\left(\tan(x^3)\right)^{-\frac{1}{2}} \cdot \sec^2(x^3) \cdot \frac{d}{dx}x^3$$

$$= \frac{1}{2}\left(\tan(x^3)\right)^{-\frac{1}{2}} \cdot \sec^2(x^3) \cdot 3x^2$$

$$= \boxed{\frac{3x^2\sec^2(x^3)}{2\sqrt{\tan(x^3)}}}$$

5. Given the implicit equation

$$\sqrt{xy} = x + y$$

Find $\frac{dy}{dx}$.

Chain rule.

$$\boxed{\frac{d}{dx}\sqrt{xy}} = \frac{d}{dx}x + \frac{d}{dx}y$$

product rule

$$\frac{1}{2}(xy)^{-\frac{1}{2}} \cdot \boxed{\frac{d}{dx}[xy]} = 1 + y'$$

$$\frac{1}{2}(xy)^{-\frac{1}{2}}\left(x \cdot \frac{d}{dx}y + y\frac{d}{dx}x\right) = 1 + y'$$

$$\frac{1}{2}(xy)^{-\frac{1}{2}}\left(\underline{xy'} + y \cdot 1\right) = 1 + \underline{y'}$$

collect on one side

dist law

$$\underline{\frac{1}{2}(xy)^{-\frac{1}{2}}xy'} + \underline{\frac{1}{2}(xy)^{-\frac{1}{2}}y} \; \underline{-y'} = 1$$

$$\frac{1}{2}(xy)^{-\frac{1}{2}}xy' - y' = 1 - \frac{1}{2}(xy)^{-\frac{1}{2}}y$$

GCF

$$y'\left(\frac{1}{2}(xy)^{-\frac{1}{2}}x - 1\right) = 1 - \frac{1}{2}(xy)^{-\frac{1}{2}}y$$

isolate y'

get rid of compound fraction

$$y' = \frac{1 - \frac{1}{2}(xy)^{-\frac{1}{2}}y}{\frac{1}{2}(xy)^{-\frac{1}{2}}x - 1} = \frac{1 - \frac{1}{2\sqrt{xy}}y}{\frac{1}{2\sqrt{xy}}x - 1} \cdot \frac{2\sqrt{xy}}{2\sqrt{xy}}$$

frac law 1

$$= \frac{\left(1 - \frac{1}{2\sqrt{xy}}y\right)2\sqrt{xy}}{\left(\frac{1}{2\sqrt{xy}}x - 1\right)2\sqrt{xy}}$$

dist

$$= \frac{2\sqrt{xy} - \frac{1}{\cancel{2\sqrt{xy}}}y\,\cancel{2\sqrt{xy}}}{\frac{1}{\cancel{2\sqrt{xy}}}x \cdot \cancel{2\sqrt{xy}} - 2\sqrt{xy}}$$

$$= \frac{2\sqrt{xy} - y}{x - 2\sqrt{xy}}$$

$$\boxed{y' = \frac{2\sqrt{xy} - y}{x - 2\sqrt{xy}}}$$

6. The kinetic energy of an object is $K = \frac{1}{2}mv^2$. If the object is accelerating at a rate of 9.8 m/s and the mass is 30 kilograms, how fast is the kinetic energy increasing when the speed is 30 meters per second?

① K and v are functions of time, m is not because mass does not change over time.

② acceleration or $\frac{dv}{dt} = 9.8\ m/s^2$

$m = 30\ kg$

$v = 30\ m/s$

need to find: $\frac{dK}{dt}$

③ $K = \frac{1}{2} m v^2$

④ $\frac{d}{dt} K = \frac{d}{dt}\left[\frac{1}{2} m v^2\right]$

$\frac{dK}{dt} = \frac{1}{2} m \frac{d}{dt}[v^2]$

$\frac{dK}{dt} = \frac{1}{2} m \cdot 2v \cdot \frac{dv}{dt} = mv\frac{dv}{dt}$

⑤ no need

⑥ $\frac{dK}{dt} = mv\frac{dv}{dt} = 30\ kg \cdot 30 \frac{m}{s} \cdot 9.8 \frac{m}{s^2}$

$= 90 \cdot 9.8 \frac{kg \cdot m^2}{s^3}$

$= \boxed{882 \frac{kg \cdot m^2}{s^3}}$

$$\begin{array}{r} 90.0 \\ \times\ 9.8 \\ \hline 72.00 \\ 810.0 \\ \hline 882 \end{array}$$